$\mathbb{F}_1$-geometry and its applications

Oliver Lorscheid (IMPA)

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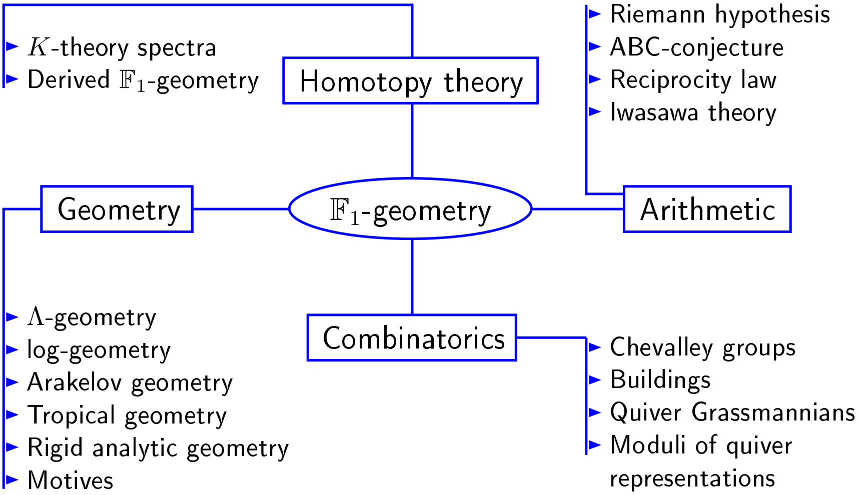
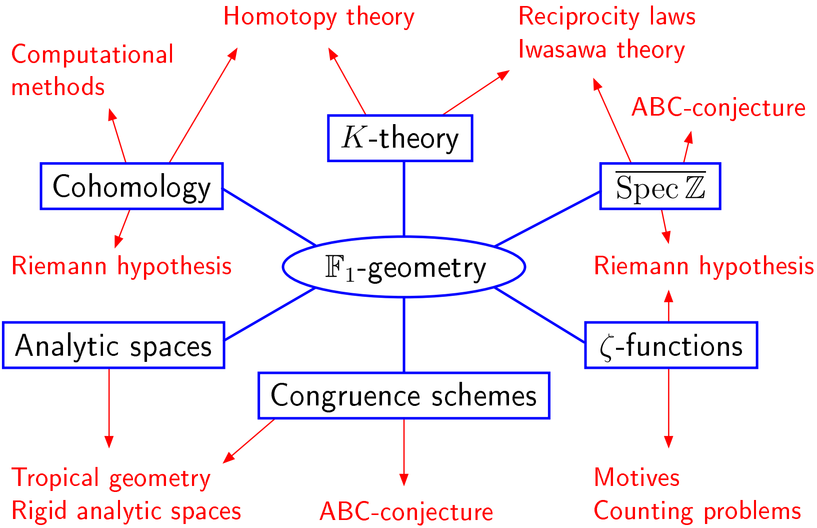
Applications



The tools



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Blueprints

Definition

A *blueprint* is a commutative monoid A together with a *pre-addition* $\mathscr{R} = \{\sum a_i \equiv \sum b_j | a_i, b_j \in A\}$, which is a set that satisfies[1]

1. $\mathscr{R}$ is an equivalence relation on $\mathbb{N}[A] = \{\sum a_i | a_i \in A\}$, and
2. $\mathscr{R}$ is additive and multiplicative, i.e. if $\sum a_i \equiv \sum b_j$ and $\sum c_k \equiv \sum d_l$, then $\sum a_i + \sum c_k \equiv \sum b_j + \sum d_l$ and $\sum a_i c_k \equiv \sum b_j d_l$.

Remark

Axioms 1 and 2 are equivalent to the existence of the quotient $B^+ = \mathbb{N}[A]/\mathscr{R}$ as a semiring.

We write $B = A /\!\!/ \mathscr{R}$, and $a \in B$ for $a \in A$.
Given a set $S = \{\sum a_i \equiv \sum b_j\}$, we denote the smallest pre-addition containing S by $\mathscr{R} = \langle S \rangle$.

[1] Sometimes a blueprint is assumed to satisfy additional axioms. For the sake of a simplified presentation, we allow ourselves to be slightly unprecise here.

Examples

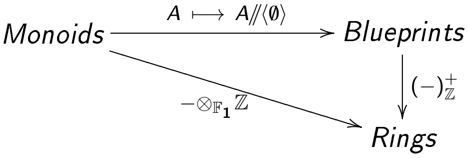
Monoids:
A commutative monoid A defines the blueprint $B = A /\!\!/ \langle \emptyset \rangle$.

Semirings:
A commutative semiring R defines the blueprint $B = R^{\bullet} /\!\!/ \mathscr{R}$ where $R^{\bullet}$ is the underlying monoid of R and
$\mathscr{R} = \{\sum a_i \equiv \sum b_j | \sum a_i = \sum b_j \text{ in } R\}$.

Universal ring $B_{\mathbb{Z}}^{+}$:
Given a blueprint $B = A /\!\!/ \mathscr{R}$, we can define the universal ring

$$B_{\mathbb{Z}}^{+} \;=\; \mathbb{Z}[A] \;/\; \left\{ \sum a_i - \sum b_j \;\middle|\; \sum a_i \equiv \sum b_j \text{ in } B \right\}.$$

We obtain a commutative diagram



Examples

Special linear group:
Define the blueprint

$$\mathbb{F}_1[\mathrm{SL}_2] \quad = \quad \mathbb{F}_1[T_1, T_2, T_3, T_4] /\!\!/ \langle T_1 T_4 \equiv T_2 T_3 + 1 \rangle$$

where

$$\mathbb{F}_1[T_1, T_2, T_3, T_4] \quad = \quad \{ T_1^{n_1} T_2^{n_2} T_3^{n_3} T_4^{n_4} | n_i \geq 0 \}$$

is the monoid of all monomials in the T_i.

Then $\mathbb{F}_1[\mathrm{SL}_2]^+_{\mathbb{Z}} = \mathbb{Z}[\mathrm{SL}_2]$ is the coordinate ring of the Chevalley group scheme $\mathrm{SL}_{2,\mathbb{Z}}$.

Blue schemes

There are straight forward generalizations of the following notions from rings and monoids to blueprints:

- prime ideals
- localizations
- the spectrum of a blueprint
- locally blueprinted spaces
- blue schemes

The category of blue schemes contains usual schemes, $\mathbb{F}_1$-schemes (after Deitmar) and objects like $\mathrm{SL}_{2,\mathbb{F}_1} = \mathrm{Spec}\,\mathbb{F}_1[\mathrm{SL}_2]$ or semiring schemes.

$\overline{\operatorname{Spec}\mathbb{Z}}$

We can define the "compactification" $\overline{\operatorname{Spec}\mathbb{Z}}$ of $\operatorname{Spec}\mathbb{Z}$ as the following locally blueprinted space $(X, \mathscr{O}_X)$.

The points $p \in X$ correspond to the places $|\ |_p$ of $\mathbb{Q}$ (if p is a finite prime or ∞) and to the discrete norm $|\ |_0$ (if $p = 0$). The points $p > 0$ are closed, and 0 is the generic point of X. For a non-empty open subset U of X, we define

$$\mathscr{O}_X(U) \;=\; \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; \left|\frac{a}{b}\right|_p \leq 1 \text{ for all } p \in U \right\} /\!/ \langle 1 + (-1) \equiv 0 \rangle.$$

Theorem (L.)

The arithmetic line $\overline{\operatorname{Spec}\mathbb{Z}}$ is 1-dimensional, while the arithmetic surface $\overline{\operatorname{Spec}\mathbb{Z}} \otimes_{\mathbb{F}_1} \overline{\operatorname{Spec}\mathbb{Z}}$ is 2-dimensional.

K-theory

There is a straight forward definition of a vector bundle over a blue scheme X as a locally free sheaf. The notion of short exact sequences turns the category $\operatorname{Bun} X$ into a quasi-exact category.

Theorem (Chu–L.–Santhanam, 2012)

The associated spectrum $\mathscr{K}(X) = \Omega|S_\bullet \operatorname{Bun}(X)|$ is a symmetric ring spectrum.

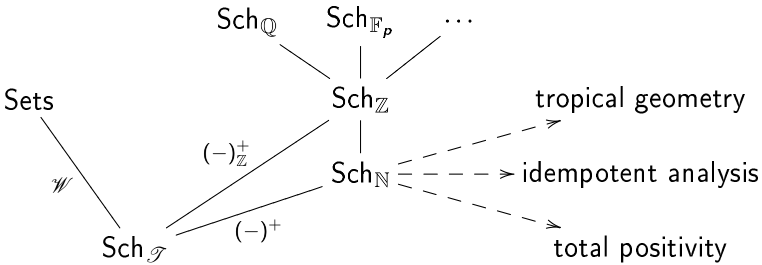
The K-theory of X is defined as $K_i(X) = \pi_i^{\mathrm{st}}(\mathscr{K}(X))$.

Theorem (Folklore, Deitmar, Chu–L.–Santhanam)

The symmetric ring spectrum $\mathscr{K}(\mathbb{F}_1)$ is weakly homotopy equivalent to the sphere spectrum $\mathbb{S}^0$. This induces a ring isomorphism $K_*(\mathbb{F}_1) \simeq \pi_*^{\mathrm{st}}(\mathbb{S}^0)$.

The Tits category

One can endow blue schemes with the class of *Tits morphisms*, which defines the *Tits category* $\mathrm{Sch}_{\mathscr{T}}$. It comes together with certain base extensions



where $\mathscr{W} : \mathrm{Sch}_{\mathscr{T}} \to \mathrm{Sets}$ is called the *Weyl extension*.

All base extensions send group objects (resp. monoids) to group objects (resp. monoids).

Tits-Weyl models

Definition

Let $\mathscr{G}$ be a Chevalley group scheme with Weyl group W. A *Tits-Weyl model* of $\mathscr{G}$ is a monoid G in $\mathrm{Sch}_{\mathscr{T}}$ such that

1. $G_{\mathbb{Z}}^{+}$ is isomorphic to $\mathscr{G}$ as a group scheme,
2. $\mathscr{W}(G)$ is isomorphic to W as a group, and
3. a certain compatibility condition is satisfied.

Theorem (L., 2012)

Let $\mathscr{G}$ be one of the following:

- $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{Sp}(2n)$, $SO(2n+1)$, $\mathrm{SO}(2n)$,
- *an adjoint Chevalley group scheme, or*
- *a split Levi subgroup of one of the above.*

Then $\mathscr{G}$ has a Tits-Weyl model.

Total positivity

For all $I, J \subset \{1, \dots, n\}$ with $\#I = \#J$, we can consider the minor

$$\Delta_{I,J}(T_{ij}) \quad = \quad \det\bigl(T_{ij} | i \in I, j \in J\bigr),$$

as an element of $\mathbb{Z}[\mathrm{SL}_n] = \mathbb{Z}[T_{ij} | i,j = 1, \dots, n] / \bigl(\det(T_{ij}) - 1\bigr)$.
Since the set of all minors generate $\mathbb{Z}[\mathrm{SL}_n]$, we have

$$\mathbb{Z}[\mathrm{SL}_n] \; = \; \mathbb{Z}[\Delta_{I,J} | I, J \subset \{1, \dots, n\}] \; / \; (\text{relations between the } \Delta_{I,J}).$$

These relations define a pre-addition $\mathscr{R}$ on the monoid $\mathbb{F}_1[\Delta_{I,J}]$, and thus a blueprint $\mathbb{F}_1[\mathrm{SL}_n] = \mathbb{F}_1[\Delta_{I,J}] /\!\!/ \mathscr{R}$.

Theorem (López Peña–L.–Reineke, work in progress)

The blue scheme $\mathrm{SL}_{n,\mathbb{F}_1} = \operatorname{Spec} \mathbb{F}_1[\mathrm{SL}_n]$ *has the unique structure of a Tits-Weyl model. It satisfies that* $\mathrm{SL}_{n,\mathbb{F}_1}(\mathbb{R}_{\geq 0})$ *is the semigroup of all totally nonnegative matrices (in the sense of Fomin-Zelevinsky).*

Quiver Grassmannians

$$k^{d_1} \xrightarrow{f_\alpha} k^{d_2} \underset{f_\gamma}{\overset{f_\beta}{\rightrightarrows}} k^{d_3} \qquad M$$

$$Q \qquad 1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3$$

Let k be a ring. A *quiver* is a finite directed graph Q. A *quiver representation* M *over* k consists of a free k-module k^{d_i} for every vertex i of Q and a linear map $f_\alpha : k^{d_i} \to k^{d_j}$ for every arrow $\alpha : i \to j$ in Q. Let $\underline{d} = (d_i)_{i \in Q}$ be the dimension vector.

For $\underline{e} = (e_i)_{i \in Q}$ with $0 \leq e_i \leq d_i$, we define the *quiver Grassmannian*

$$\mathrm{Gr}_{\underline{e}}(M, k) \quad = \quad \{ \text{ subrepresentations } N \subset M \mid \underline{\dim} N = \underline{e} \},$$

which turns out to be the set of k-rational points of a projective k-scheme $\mathrm{Gr}_{\underline{e}}(M)_k$.

Theorem (Reineke, 2012)

Every projective variety over k can be represented as a quiver Grassmannian.

$\mathbb{F}_1$-points of quiver Grassmannians

Let $k = \mathbb{Z}$. Denote the standard basis vectors of $\mathbb{Z}^{d_i}$ by $e_{i,r}$. The set ${}^*\mathrm{Gr}_{\underline{e}}(M,\mathbb{F}_1)^*$ of "*$\mathbb{F}_1$-rational points*" *of* $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{Z}}$ is the set of all subrepresentations $N \subset M$ of dimension $\underline{\dim} N = \underline{e}$ such that

1. N_i is spanned by $\{e_{i,r}\} \cap N_i$ for every $i \in Q$, and
2. $f_\alpha(e_{i,r}) \in \{e_{j,s}\} \cup \{0\}$ for all $\alpha : i \to j$ and $e_{i,r} \in N_i$.

Theorem (L.)

There is a canonical blue scheme $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}$ *of finite type over* $\mathbb{F}_1$ *such that* $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{Z}} = \left(\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}\right)^+_{\mathbb{Z}}$. *There is a canonical inclusion*

$$\iota : \ {}^*\mathrm{Gr}_{\underline{e}}(M,\mathbb{F}_1)^* \quad \hookrightarrow \quad \mathscr{W}(\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}),$$

which is a bijection if $\# f_\alpha^{-1}(e_{j,s}) \leq 1$ *for all* $\alpha : i \to j$ *and* $e_{j,s} \in N_j$. *If furthermore* Q *is acyclic, then the Euler characteristic of* $\mathrm{Gr}_{\underline{e}}(M,\mathbb{C})$ *equals* $\#\mathscr{W}(\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1})$ *(by a result of Cerulli-Irelli).*