Tits's dream: buildings over $\mathbb{F}_1$ and combinatorial flag varieties

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Based on joint works with
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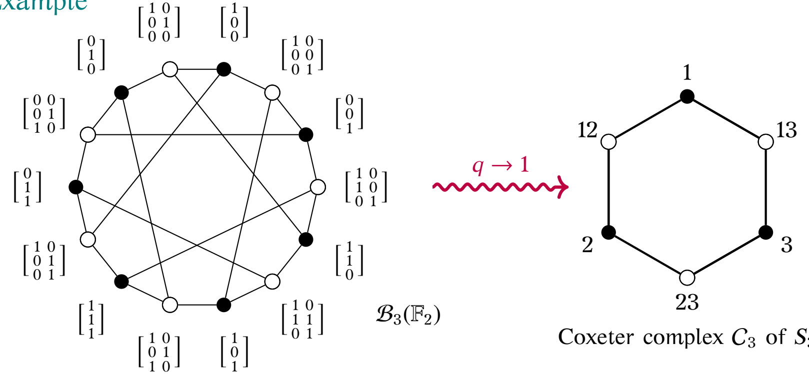
Motivation

Jacques Tits, 1956

The Coxeter complex C_n of S_n behaves like the limit $q \to 1$ of the building $\mathcal{B}_n(\mathbb{F}_q)$ of flags of linear subspaces of $\mathbb{F}_q^n$.

Example



$\mathcal{B}_3(\mathbb{F}_2)$

Coxeter complex C_3 of S_3

$$\begin{array}{ccccccc} \#\{\text{b/w vertices}\} & = & q^2 + q^1 + q^0 & \xrightarrow{q\to 1} & 1^2 + 1^1 + 1^0 & = & \#\{\text{b/w vertices}\} \\ \text{valency} & = & q^1 + q^0 & \longrightarrow & 1^1 + 1^0 & = & \text{valency} \end{array}$$

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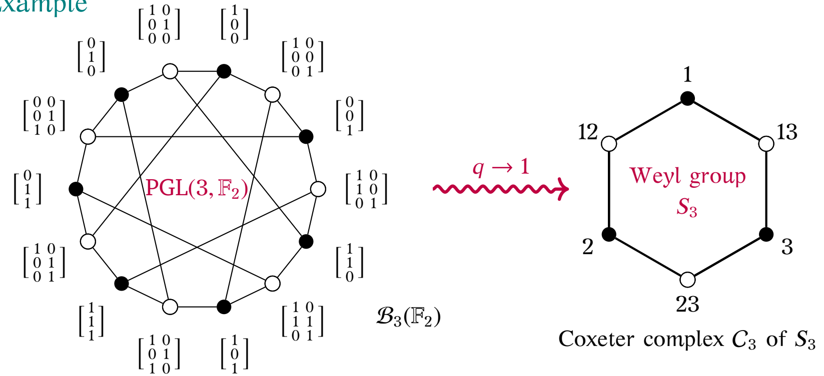


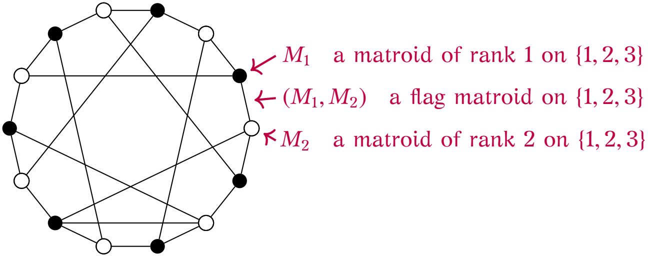
Coxeter complex C_3 of S_3

The limit $q \to 1$ extends to the symmetry groups (in terms of their invariants).

Borovik, Gelfand and White, 2003

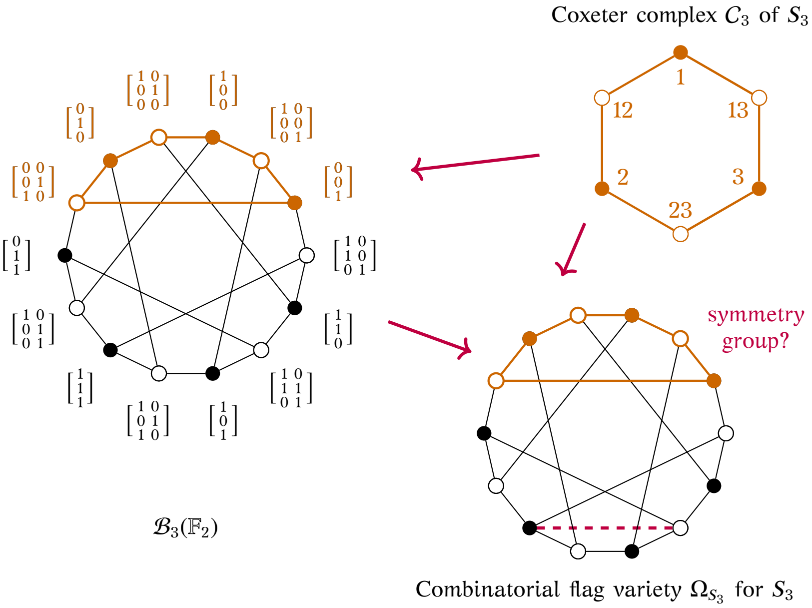
> *The Coxeter complex has a relatively poor structure. In many aspects, combinatorial flag varieties are more suitable candidates for the role of a "universal" combinatorial geometry over the field of* 1 *element.*

The combinatorial flag variety Ω_{S_n} is the simplicial complex whose simplices consist of flag matroids on $\{1, \ldots, n\}$.



Combinatorial flag variety Ω_{S_3} for S_3

Maps between the simplicial complexes



I. Bands

Bands

A **ring** is a commutative semigroup $(R, \cdot)$ with constants 0, 1 and -1, together with an addition $+ : R \times R \to R$ such that...

A **band** is a commutative semigroup $(B, \cdot)$ with constants 0, 1 and -1, together with a set $N_B = \{\sum a_i \mid a_1, \dots, a_n \in B\}$ of *zero sums* such that for all $a, b \in B$:

1. $0 \cdot a = 0$ and $1 \cdot a = a$; (*constants*)
2. $0 \in N_B, \quad B \cdot N_B = N_B, \quad N_B + N_B = N_B$; (*ideal property*)
3. $a + b \in N_B$ if and only if $b = -a := (-1) \cdot a$. (*inverses*)

A **band morphism** is multiplicative map $B_1 \to B_2$ that preserves the constants and zero sums. This defines the category Bands.

Example: A ring R is a band with

$$N_R \;=\; \{\sum a_i \mid \sum a_i = 0 \text{ as sum in } R\}.$$

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Other examples:

- The **Krasner hyperfield** $\mathbb{K} = \{0,\ 1 = -1\}$ with

$$N_{\mathbb{K}} = \{0, \quad 1 + 1, \quad 1 + 1 + 1, \dots\}.$$

- The **regular partial field** $\mathbb{F}_1^{\pm} = \{0,\ 1,\ -1\}$ with (*initial*)

$$N_{\mathbb{F}_1^{\pm}} = \{0, \quad 1 - 1, \quad 1 - 1 + 1 - 1, \quad 1 - 1 + \cdots + 1 - 1, \dots\}.$$

Band schemes

An **affine band scheme** is a representable functor $\mathrm{Hom}(C, -) : \mathrm{Bands} \to \mathrm{Sets}$.

Example: The **affine** ***n*****-space** $\mathbb{A}^n : \mathrm{Bands} \to \mathrm{Sets}$ is defined as $\mathbb{A}^n(B) = B^n$. It is an affine band scheme, represented by the *free algebra* $C = \mathbb{F}_1^{\pm}[T_1, \dots, T_n]$ over the regular partial field $\mathbb{F}_1^{\pm}$.

A **band scheme** is a functor $X : \mathrm{Bands} \to \mathrm{Sets}$ that has an *open cover* by affine band schemes.

Example: The **projective** ***n*****-space** $\mathbb{P}^n : \mathrm{Bands} \to \mathrm{Sets}$ is defined as

$$\mathbb{P}^n(B) \;=\; \{(a_0, \dots, a_n) \in B^{n+1} \mid a_i \in B^\times \text{ for some } i\} \;/\; B^\times.$$

A **morphism** of band schemes is a morphism of functors.

II. Flags

Grassmannians

Let $0 \leq r \leq n$ and $E = \{1, \dots, n\}$. Let $\binom{E}{r}$ denote the collection of r-subsets I of E.

The **Grassmannian** $\mathrm{Gr}(r,n) : \mathrm{Bands} \to \mathrm{Sets}$ sends a band B to the subset

$$\mathrm{Gr}(r,n)(B) \quad \text{of} \quad \mathbb{P}^{\binom{n}{r}-1}(B)$$

that consists of all $[\Delta_I]_{I \in \binom{E}{r}}$ that satisfy the *Plücker relations*

$$\sum_{j \in J-I} (-1)^{\epsilon(I,j)} \; \Delta_{I \cup \{j\}} \; \Delta_{J-\{j\}} \quad \in \quad N_B$$

for all $I, J \subset E$ with $\#I = r-1, \quad \#J = r+1$ and

$$\epsilon(I,j) \;=\; \#\{i \in I \,\big|\, i < j\}.$$

Flag varieties

Let $0 < r_1 < \cdots < r_s < n$ and $\mathbf{r} = (r_1, \ldots, r_s)$.

The **flag variety** $\mathrm{Fl}(\mathbf{r}, n) : \mathrm{Bands} \to \mathrm{Sets}$ sends a band B to the subset

$$\mathrm{Fl}(\mathbf{r}, n)(B) \quad \text{of} \quad \prod_{i=1}^{s} \mathrm{Gr}(r_i, n)(B)$$

that consists of all $[\Delta_{i,I}]_{I \in \binom{E}{r_i}}$ that satisfy the *incidence relations*

$$\sum_{j \in J - I} (-1)^{\epsilon(I,j)} \; \Delta_{k, I \cup \{j\}} \; \Delta_{l, J - \{j\}} \quad \in \quad N_B$$

for all $1 \leq k \leq l \leq s, \quad I, J \subset E$ with $\#I = r_k - 1, \quad \#J = r_l + 1$ and

$$\epsilon(I, j) \;=\; \#\{i \in I \,\big|\, i < j\}.$$

Rational point sets

For a field K, we recover the usual bijections

$$\mathrm{Gr}(r,n)(K) \longrightarrow \Big\{ \text{linear subspaces } V \text{ of } K^n \text{ of dimension } r \Big\}$$

and

$$\mathrm{Fl}(\mathbf{r},n)(K) \longrightarrow \left\{ \begin{array}{c} \text{flags } V_1 \subset \cdots \subset V_s \text{ of linear subspaces} \\ \text{of } K^n \text{ of dimensions } \dim V_i = r_i \end{array} \right\}.$$

Rational point sets

Let $\mathbb{K} = \{0,1\}$ with $N_{\mathbb{K}} = \{n.1 \mid n \neq 1\}$ be the Krasner hyperfield.

Theorem (Baker-L '21)

The map

$$\begin{array}{ccc} \mathrm{Gr}(r,n)(\mathbb{K}) & \longrightarrow & \left\{ \textit{matroids } M \textit{ of rank } \mathrm{rk}\, M = r \textit{ on } E \right\} \\ [\Delta_I] & \longmapsto & \mathcal{B}(M) = \{ I \in \binom{E}{r} \,\big|\, \Delta_I \neq 0 \} \quad \textit{(bases)} \end{array}$$

is a bijection.

Theorem (Jarra-L '24)

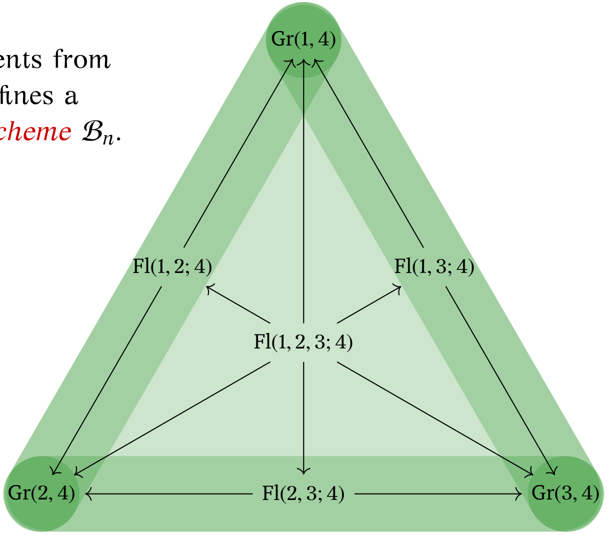
The above map extends to a bijection

$$\mathrm{Fl}(\mathbf{r},n)(\mathbb{K}) \quad \longrightarrow \quad \left\{ \begin{array}{c} \textit{flags } (M_1, \ldots, M_s) \textit{ of matroids} \\ \textit{of ranks } \mathrm{rk}\, M_i = r_i \textit{ on } E \end{array} \right\}.$$

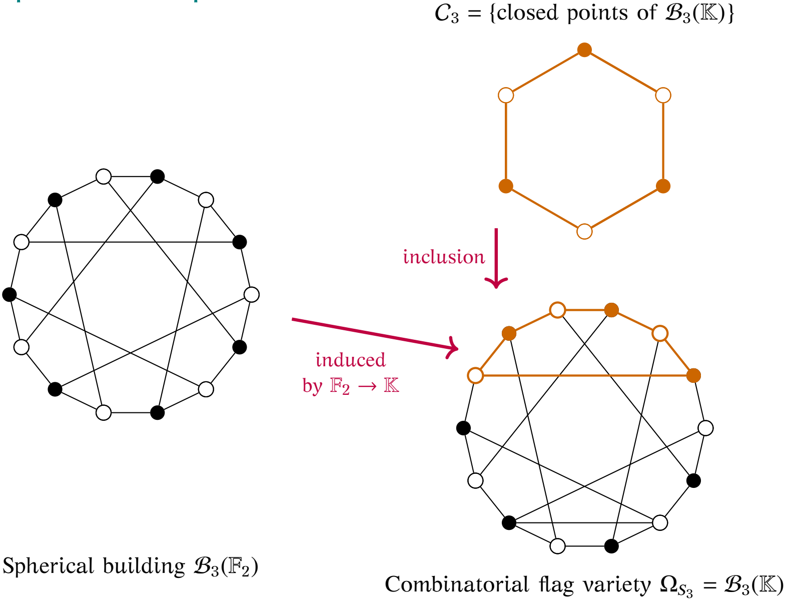
Simplicial band schemes

Omitting coefficients from $\mathbf{r} = (r_1, \ldots, r_s)$ defines a *simplicial band scheme* $\mathcal{B}_n$.

For example $\mathcal{B}_4$:



Simplicial complexes revisited



III. Crowds

Crowds

A **crowd** is a set G together with an *identity* 1 and a **crowd law**, which is a subset $R \subset G^3$, such that for all $a, b, c \in G$

1. $(a, 1, 1) \in R$ if and only if $a = 1$;
2. $(b, a, 1) \in R$ if $(a, b, 1) \in R$;
3. $(c, a, b) \in R$ if $(a, b, c) \in R$.

A **crowd morphism** is a map $f : G_1 \to G_2$ such that $f(1) = 1$ and $(f(a), f(b), f(c)) \in R_2$ for all $(a, b, c) \in R_1$. This defines the category Crowds.

Example

A group G with identity 1 is a crowd with respect to the crowd law

$$R = \left\{(a, b, c) \in G^3 \,\middle|\, abc = 1\right\}.$$

This defines a fully faithful embedding Groups $\to$ Crowds.

Algebraic crowds

The category of crowds comes with two functors

$$\mathcal{F}_G : \text{Crowds} \to \text{Sets} \qquad \text{and} \qquad \mathcal{F}_R : \text{Crowds} \to \text{Sets},$$

which send a crowd G to its underlying set $\mathcal{F}_G(G) = G$ and to its crowd law $\mathcal{F}_R(G) = R$, respectively.

An **algebraic crowd** is a functor $\mathcal{G} : \text{Bands} \to \text{Crowds}$ for which both $\mathcal{F}_G \circ \mathcal{G} : \text{Bands} \to \text{Sets}$ and $\mathcal{F}_R \circ \mathcal{G} : \text{Bands} \to \text{Sets}$ are band schemes.

The special linear crowd

The algebraic crowd $\mathrm{SL}_2 : \mathrm{Bands} \to \mathrm{Crowds}$ is defined as

$$\mathrm{SL}_2(B) \;=\; \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in B^4 \,\middle|\, ad - bc - 1 \in N_B \right\}$$

with identity $\mathbf{1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and crowd law R that consists of all $(a^{(1)}, a^{(2)}, a^{(3)}) \in \mathrm{SL}_2(B)^3$ such that

$$\sum_{k,l=1}^{n} a_{i,k}^{(\sigma(1))} \cdot a_{k,l}^{(\sigma(2))} \cdot a_{l,j}^{(\sigma(3))} \quad - \quad \mathbf{1}_{i,j} \quad \in \quad N_B$$

for every $i, j \in \{1, 2\}$ and $\sigma \in A_3$.

- For a ring R, the crowd $\mathrm{SL}_2(R)$ is the usual special linear group of 2×2-matrices over R.
- $\mathrm{SL}_2(\mathbb{F}_1^{\pm})$ is the *subcrowd* of $\mathrm{SL}_2(\mathbb{Z})$ that consists of all matrices with coefficients in $\mathbb{F}_1^{\pm} = \{0, 1, -1\}$.
- $\mathrm{SL}_n : \mathrm{Bands} \to \mathrm{Crowds}$ is defined analogous.

$\mathbb{K}$-points of the special linear crowd

$\mathrm{SL}_2(\mathbb{K})$ consists of the seven elements

$$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right],\quad \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right],\quad \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right],\quad \left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right],\quad \left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right],\quad \left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right],\quad \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right].$$

Its crowd law is *multi-valued*: it contains triples (a, b, c) and (a, b, d) with $c \neq d$, such as

$$\Big(\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right],\ \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right],\ \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\Big) \qquad \text{and} \qquad \Big(\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right],\ \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right],\ \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\Big).$$

In particular, $\mathrm{SL}_2(\mathbb{K})$ is *not* a group.

Finding the expected $\mathbb{F}_1$-points:

The Weyl group "$\mathrm{SL}_n(\mathbb{F}_1)$" $= S_n$ of SL_n appears naturally as the subcrowd of permutation matrices in $\mathrm{SL}_n(\mathbb{K})$.

Crowd activity

Observation: A group action $G \times X \to X$ is determined by its graph T in $G \times X \times X$.

A **crowd activity** of an algebraic crowd $\mathcal{G}$ on a band schemes $\mathcal{X}$ is a subscheme $\mathcal{T}$ of $\mathcal{G} \times \mathcal{X} \times \mathcal{X}$.

There is a natural crowd activity of SL_n on $\mathrm{Gr}(r,n)$ and $\mathrm{Fl}(\mathbf{r},n)$, which extends to an activity of SL_n on $\mathcal{B}_n$.

- Taking $\mathbb{F}_q$-rational points recovers the usual group action of $\mathrm{SL}_n(\mathbb{F}_q)$ on the building $\mathcal{B}_n(\mathbb{F}_q)$.
- Taking $\mathbb{K}$-rational points exhibits a crowd activity of $\mathrm{SL}_n(\mathbb{K})$ on the combinatorial flag variety $\Omega_{S_n} = \mathcal{B}_n(\mathbb{K})$.
- The respective subspaces of closed points recovers the action of S_n on its Coxeter complex C_n.

Simplicial complexes and their symmetries