Categories of matroids and matroid bundles

Oliver Lorscheid
IMPA / University of Groningen

Based on joint ideas
with Matthew Baker, Manoel Jarra and Tong Jin

I. Bands (joint with Baker and Jin)

Pointed monoids

A **pointed monoid** is a commutative semigroup A with 1 and 0, i.e. $1 \cdot a = 1$ and $0 \cdot a = 0$ for all $a \in A$.

The **unit group** of A is

$$A^\times \;=\; \{a \in A \,\big|\, ab = 1 \text{ for some } b \in A\}.$$

The **ambient semiring** of A is

$$A^+ \;=\; \mathbb{N}[A]/\langle 0_A \sim 0_{\mathbb{N}[A]} \rangle \;=\; \{\textstyle\sum a_i \,\big|\, a_i \in A - \{0\}\}.$$

An **ideal** of A^+ is a subset I such that $0 \in I$, $\quad I + I = I$ and $A^+ \cdot I = I$.

Bands

A **band** is a pointed monoid B together with an ideal $N_B \subset B^+$ (the *nullset*) such that every $a \in B$ has a unique $-a \in B$ (its *additive inverse*) with $a + (-a) \in N_B$.

A **band morphism** is a multiplicative map $f : B \to C$ with $f(0) = 0$ and $f(1) = 1$ such that $\sum f(a_i) \in N_C$ for all $\sum a_i \in N_B$.

This defines the category Bands.

An **idyll** is a band B with $B^\times = B - \{0\}$.

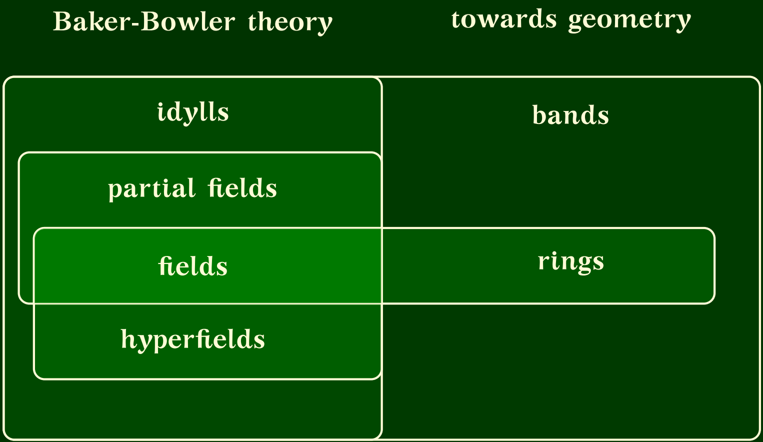
Examples

- A ring R defines the band $B = (R, \cdot)$ with nullset
$$N_B = \{ \sum a_i \,\big|\, \sum a_i = 0 \text{ in } R \}.$$
If R is a field, then B is an idyll.

- Other examples of idylls:

$\mathbb{K} = \{0, 1\}$	$N_{\mathbb{K}} = \{n.1 \mid n \neq 1\}$	*Krasner hyperfield*
$\mathbb{S} = \{0, \pm 1\}$	$N_{\mathbb{S}} = \{n.1 + m.(-1) \mid n = m = 0 \text{ or } n \neq 0 \neq m\}$	*sign hyperfield*
$\mathbb{T} = \mathbb{R}_{\geq 0}$	$N_{\mathbb{T}} = \{\sum a_i \mid \text{maximum occurs twice}\}$	*tropical hyperfield*
$\mathbb{F}_1^{\pm} = \{0, \pm 1\}$	$N_{\mathbb{F}_1^{\pm}} = \{n.1 + n.(-1) \mid n \geq 0\}$	*regular partial field*

Categorical landscape



II. Baker-Bowler theory

B-matroids

Let $E = \{1, \ldots, n\}$ and $0 \leq r \leq n$. Let B be an idyll.

A **Grassmann-Plücker function** (of rank r on E) in B is a non-trivial and alternating map $\Delta : E^r \to B$ that satisfies the *Plücker relations*

$$\sum_{k=0}^{r} (-1)^k \, \Delta(e_0, \ldots, \widehat{e_k}, \ldots, e_r) \, \Delta(e_k, d_2, \ldots, d_r) \in N_B$$

for all $e_0, \ldots, e_r, d_2, \ldots, d_r \in E$.

A ***B*-matroid** (of rank r on E) is a $B^\times$-class $M = [\Delta]$ of a Grassmann-Plücker function $\Delta : E^r \to B$.

Baker-Bowler theory

Example

1. A matroid M of rank r on E corresponds to the $\mathbb{K}$-matroid $[\Delta : E^r \to \mathbb{K}]$ with

$$\Delta(e_1, \ldots, e_r) = \begin{cases} 1 & \text{if } \{e_1, \ldots, e_r\} \text{ is a basis of } M; \\ 0 & \text{if not.} \end{cases}$$

2. Oriented matroids correspond to $\mathbb{S}$-matroids.
3. Valuated matroids correspond to $\mathbb{T}$-matroids.

Theorem (Baker-Bowler ’19)

- *Cycles:* $\mathcal{C}(M) \subset B^n$, *with cryptomorphic axioms*
- *Duality:* M^*
- *Orthogonality:* $\mathcal{C}(M) \perp \mathcal{C}(M^*)$
- *Vectors:* $\mathcal{V}(M) = \mathcal{C}(M^*)^\perp$, *axioms by* [Anderson ’19]
- *Minors:* $M \backslash I / J$

Perfect idylls

An idyll B is **perfect** if $\mathcal{V}(M) \perp \mathcal{V}(M^*)$ for every B-matroid M.

Example

1. All (partial) fields are perfect.
2. $\mathbb{K}$, $\mathbb{S}$ and $\mathbb{T}$ are perfect.
3. There are many non-perfect idylls.

III. Morphisms
(joint with Baker and Jarra)

$\mathbb{F}_1$-linear maps and pointed B-matroids

Let $E_i = \{0, 1, \ldots, n_i\}$ for $i = 1, 2$.

A map $f : E_1 \to E_2$ is **$\mathbb{F}_1$-linear** if $f(0) = 0$ and if $\# f^{-1}(d) \leq 1$ for all $d \in E_2 - \{0\}$.

The **adjoint** of f is

$$\begin{array}{rccl} f^{\#}: & E_2 & \longrightarrow & E_1 \\ & d & \longmapsto & \begin{cases} e & \text{if } f^{-1}(d) = \{e\}, \\ 0 & \text{if } f^{-1}(d) = \emptyset \text{ or } d = 0. \end{cases} \end{array}$$

Let B an idyll. A **pointed B-matroid** is a B-matroid $M = [\Delta : E^r \to B]$ for which 0 is a *loop*, i.e. $\Delta(0, e_2, \ldots, e_r) = 0$ for all $e_2, \ldots, e_r \in E$.

Morphisms over perfect idylls

Assume that B is perfect. Let $M_i = [\Delta_i : E_i^{r_i} \to B]$ be pointed B-matroids for $i = 1, 2$.

A **morphism** $M_1 \to M_2$ is a *strong* $\mathbb{F}_1$-linear map $f : E_1 \to E_2$, i.e. $f^* : B^{E_2} \to B^{E_1}$ restricts to $\mathcal{V}(M_2^*) \to \mathcal{V}(M_1^*)$.

This defines the category Mat_B of pointed B-matroids.

Theorem

Let $f : E_1 \to E_2$ an $\mathbb{F}_1$-linear map. Then the following are equivalent:

1. *f defines a morphism $M_1 \to M_2$.*
2. *$f^\# : E_2 \to E_1$ defines a morphism $M_2^* \to M_1^*$.*
3. *$f^*(\mathcal{C}(M_2^*)) \perp \mathcal{C}(M_1)$.*
4. *For all $e_0, \ldots, e_{r_1} \in E_1$ and $d_2, \ldots, d_{r_2} \in E_2$,*

$$\sum_{k=0}^{r_1} (-1)^k \, \Delta_1(e_0, \ldots, \widehat{e_k}, \ldots, e_{r_1}) \, \Delta_2(f(e_k), d_2, \ldots, d_{r_2}) \;\in\; N_B.$$

Examples

Let M be a pointed B-matroid and S a pointed subset of M.

1. The *restriction* $M|_S \to M$ is a morphism.
2. The *contraction* $M \to M/S$ is a morphism.
3. Matroid *quotients* $M \to N$ are morphisms.
4. Identifying parallel elements defines a strong map $M \to N$ that is *not* $\mathbb{F}_1$-linear.

Perfection

Remark

If B is not perfect, then restrictions and contractions fail to preserve vectors in general.

Let B be an idyll. The **perfection** of B is the limit $B^{\text{perf}} = \lim P$ over all morphisms $B \to P$ into perfect idylls P.

Theorem

1. *B^{perf} is perfect.*
2. *The canonical morphism $\omega : B \to B^{\text{perf}}$ is a bijection.*
3. *$N_{B^{\text{perf}}} = \bigcap N_P$ as subsets of B^+ where N_P ranges over all bijective morphisms $B \to P$ into perfect idylls P.*

Problem

We do not know an explicit description of $N_{B^{\text{perf}}}$.

Preperfect B-matroids

We define the category of **preperfect B-matroids** as $\mathrm{Mat}_B = \mathrm{Mat}_{B^{\mathrm{perf}}}$.

Remark

Every B-matroid (in Baker-Bowler's sense) is preperfect.
Both notions agree if B is perfect.

IV. Matroid bundles (joint with Baker)

Band schemes

An **affine band scheme** is a representable functor $\mathrm{Hom}(B,-) : \mathrm{Bands} \to \mathrm{Sets}$.

A **band scheme** is a functor $X : \mathrm{Bands} \to \mathrm{Sets}$ that has an *open cover* by affine band schemes.

A **morphism** of band schemes is a morphism of functors.

Example

1. The **projective n-space** $\mathbb{P}^n : \mathrm{Bands} \to \mathrm{Sets}$ is defined as

$$\mathbb{P}^n(B) \;=\; (B^{n+1} - \{0\}) \,/\, B^\times \;=\; \{[a_0 : \cdots : a_n] \mid a_0, \ldots, a_n \in B\}.$$

2. The **Grassmannian** $\mathrm{Gr}(r,n) : \mathrm{Bands} \to \mathrm{Sets}$ is defined as

$$\mathrm{Gr}(r,n)(B) \;=\; \left\{[\Delta(e_1, \ldots, e_r)] \,\middle|\, \text{Plücker relations}\right\} \;\subset\; \mathbb{P}^{n^r-1}(B).$$

The moduli space of matroids

Let X be a band scheme. A **matroid bundle** on X is an isomorphism class of a *Grassmann-Plücker function* $\Delta : E^r \to \Gamma(X, \mathcal{L})$ where $\mathcal{L}$ is a *line bundle* on X.

Theorem

1. *For every idyll B and $X = \mathrm{Hom}(B, -)$, there is a canonical bijection*

$$\Phi_B : \{B\text{-}matroids\} \longrightarrow \{matroid\ bundles\ on\ X\}.$$

2. *For every band scheme X, there is a canonical bijection*

$$\Psi_X : \mathrm{Hom}\,(X, \mathrm{Gr}(r,n)) \longrightarrow \left\{ \begin{array}{c} matroid\ bundles\ on\ X \\ of\ rank\ r\ on\ \{1, \dots, n\} \end{array} \right\}.$$

The **universal family** on $\mathrm{Gr}(r,n)$ is $\mathcal{M}^{\mathrm{univ}} = \Psi_{\mathrm{Gr}(r,n)}(\mathrm{id}_{\mathrm{Gr}(r,n)})$.
The matroid bundle $\Psi_X(\varphi : X \to \mathrm{Gr}(r,n))$ is the *pullback* $\varphi^* \mathcal{M}^{\mathrm{univ}}$.

Morphisms

Step 1: Explicit construction of a coherent subsheaf $\mathcal{C}^{\text{univ}} = \mathcal{C}_{\mathcal{M}^{\text{univ}}}$ (the *universal cycle bundle*) of $\mathcal{O}^n_{\text{Gr}(r,n)}$.

Step 2: For a matroid bundle $\mathcal{M} = \varphi^*\mathcal{M}^{\text{univ}}$ on a band scheme X, we define its **cycle bundle** as $\mathcal{C}_{\mathcal{M}} = \varphi^*_{\mathcal{M}}\mathcal{C}^{\text{univ}}$.

Remark: This recovers Baker-Bowler's cycles of a B-matroid.

Step 3: The **vector bundle** of $\mathcal{M}$ is the coherent subsheaf $\mathcal{V}_{\mathcal{M}} = \mathcal{C}^{\perp}_{\mathcal{M}^*}$ of $\mathcal{O}^n_X$.

Problem: In general, $\mathcal{V}_{\mathcal{M}^*}$ is *not* orthogonal to $\mathcal{V}_{\mathcal{M}}$.

Step 4: Construct the **perfection** X^{perf} of a band scheme X. We have $\mathcal{V}_{\mathcal{M}^*} \perp \mathcal{V}_{\mathcal{M}}$ for all matroid bundles $\mathcal{M}$ on X^{perf}.

Step 5: A **morphism** $\mathcal{M}_1 \to \mathcal{M}_2$ of pointed matroid bundles on X^{perf} is an $\mathbb{F}_1$-linear map $f : E_1 \to E_2$ between the respective ground sets such that $f^* : \mathcal{O}^{E_2}_X \to \mathcal{O}^{E_1}_X$ maps $\mathcal{V}_{\mathcal{M}_2^*}$ to $\mathcal{V}_{\mathcal{M}_1^*}$.

V. The Tutte-Grothendieck ring

The Tutte-Grothendieck ring

The **Tutte-Grothendieck ring** $K_0^{\mathrm{mat}}(\mathbb{K})$ is the quotient of the free abelian group $\bigoplus \mathbb{Z}.[M]$ generated by all isomorphism classes $[M]$ of (pointed) matroids modulo all relations of the form

$$[M] = [M \backslash e] + [M/e]$$

where e is an element of M that is neither a loop nor a coloop. The product $[M] \cdot [N] = [M \oplus N]$ turns $K_0^{\mathrm{mat}}(\mathbb{K})$ into a ring.

Theorem (Tutte)

The map $\alpha \mapsto [U(1,1)]$ and $\beta \mapsto [U(0,1)]$ defines a ring isomorphism $\mathbb{Z}[\alpha,\beta] \to K_0^{\mathrm{mat}}(\mathbb{K})$, which maps the Tutte polynomial $T_M(\alpha,\beta)$ of a matroid M to its class $[M]$ in $K_0^{\mathrm{mat}}(\mathbb{K})$.

Algebraic K-theory

The **algebraic K-theory** $K_0^{\mathrm{alg}}(\mathbb{K})$ of $\mathrm{Mat}_{\mathbb{K}}$ is the free abelian group $\bigoplus \mathbb{Z}.[M]$ generated by all isomorphism classes $[M]$ of pointed matroids modulo all relations of the form

$$[M] = [M|_S] + [M/S] \qquad (\text{“}0 \to M|_S \to M \to M/S \to 0\text{”})$$

where S is a pointed subset of M.

Theorem (Eppolito-Jun-Szczesny)

The map $\alpha \mapsto [U(1,1)]$ and $\beta \mapsto [U(0,1)]$ defines a group isomorphism $\mathbb{Z}.\alpha \oplus \mathbb{Z}.\beta \to K_0^{\mathrm{alg}}(\mathbb{K})$.

Remark: The map

$$\deg : \ K_0^{\mathrm{mat}}(\mathbb{K}) \ = \ \mathbb{Z}[\alpha,\beta] \ \longrightarrow \ \mathbb{Z}.\alpha \oplus \mathbb{Z}.\beta \ = \ K_0^{\mathrm{alg}}(\mathbb{K})$$

with $\deg P = (\deg_\alpha P, \deg_\beta P)$ is a valuation:

$$\deg(1) = 0, \qquad \deg(P+Q) \leq \max\{\deg P, \deg Q\},$$

$$\deg(P \cdot Q) = \deg P + \deg Q, \qquad \deg([M]) = [M].$$

Matroidal K-theory

Let X be a band scheme. The **matroidal K-theory** of X is

$$K_0^{\mathrm{mat}}(X) \;=\; \bigoplus_{\substack{\text{isomorphism}\\ \text{classes } [\mathcal{M}]}} \mathbb{Z}.[\mathcal{M}] \Big/ \langle [\mathcal{M}] - [\mathcal{M}\backslash e] - [\mathcal{M}/e] \rangle$$

where $\mathcal{M}$ is a pointed matroid bundle on X and e is not a loop nor a coloop of $\mathcal{M}$. It is a ring w.r.t. $[\mathcal{M}] \cdot [\mathcal{N}] = [\mathcal{M} \oplus \mathcal{N}]$.

The **algebraic K-theory** of X is

$$K_0^{\mathrm{alg}}(X) \;=\; \bigoplus_{\substack{\text{isomorphism}\\ \text{classes } [\mathcal{M}]}} \mathbb{Z}.[\mathcal{M}] \Big/ \langle [\mathcal{M}] - [\mathcal{M}|_S] - [\mathcal{M}/S] \rangle$$

where S is a pointed subset of $\mathcal{M}$.

The association $[\mathcal{M}] \mapsto [\mathcal{M}]$ defines a valuation

$$\deg : K_0^{\mathrm{mat}}(X) \longrightarrow K_0^{\mathrm{alg}}(X).$$

The universal Tutte class

The **universal Tutte class** on $\mathrm{Gr}(r,n)$ is the class of the universal matroid bundle $\mathcal{M}^{\mathrm{univ}}$ in $K_0^{\mathrm{mat}}(\mathrm{Gr}(r,n))$.

The pullback of matroid bundles along a morphism $\varphi : Y \to X$ of band schemes defines a ring homomorphism

$$\varphi^* : K_0^{\mathrm{mat}}(X) \longrightarrow K_0^{\mathrm{mat}}(Y).$$

Theorem

Let M be a matroid and $\mathcal{M} = \varphi^* \mathcal{M}^{\mathrm{univ}}$ the corresponding matroid bundle on $\mathrm{Hom}(\mathbb{K}, -)$. Then the Tutte polynomial of M is the pullback $\varphi^*([\mathcal{M}^{\mathrm{univ}}])$ as a class of $K_0^{\mathrm{mat}}(\mathbb{K}) = \mathbb{Z}[\alpha,\beta]$.

Fink-Speyer's theorem

The morphisms of complex varieties

$$\mathrm{Gr}(r,n)_{\mathbb{C}} \xleftarrow{\pi_r} \mathrm{Fl}(1,r,n-1;n)_{\mathbb{C}} \xrightarrow{\pi_{1,n-1}} \mathbb{P}^{n-1}_{\mathbb{C}} \times (\mathbb{P}^{n-1}_{\mathbb{C}})^{\vee}$$

induce homomorphisms

$$\begin{array}{ccc} K_0(\mathrm{Gr}(r,n)_{\mathbb{C}}) & \xrightarrow{\quad\Xi\quad} & \mathbb{Z}[\alpha_0,\beta_0]/\langle\alpha_0^n,\beta_0^n\rangle \\ \downarrow{\scriptstyle \pi_r^*} & & {\scriptstyle\simeq}\uparrow \\ K_0(\mathrm{Fl}(1,r,n-1;n)_{\mathbb{C}}) & \xrightarrow{\pi_{1,n-1,*}} & K_0(\mathbb{P}^{n-1}_{\mathbb{C}} \times (\mathbb{P}^{n-1}_{\mathbb{C}})^{\vee}) \end{array}$$

where α_0 and β_0 are the classes of a hyperplane of $\mathbb{P}^{n-1}_{\mathbb{C}}$ and of $(\mathbb{P}^{n-1}_{\mathbb{C}})^{\vee}$, respectively.

A matroid M defines a natural class $c_M \in K_0(\mathrm{Gr}(r,n)_{\mathbb{C}})$.

Theorem (Fink-Speyer)

$T_M(\alpha_0,\beta_0) = \Xi(c_M \cdot [\mathcal{O}(1)])$ is the Tutte polynomial of M, considered as element of $\mathbb{Z}[\alpha_0,\beta_0]/\langle\alpha_0^n,\beta_0^n\rangle$.

Can we recover Fink-Speyer's theorem?

$$K_0(\mathrm{Gr}(r,n)_{\mathbb{C}}) \xrightarrow{\pi_r^*} K_0(\mathrm{Fl}(1,r,n-1;n)_{\mathbb{C}}) \xrightarrow{\pi_{1,n-1,*}} K_0(\mathbb{P}^{n-1}_{\mathbb{C}} \times (\mathbb{P}^{n-1}_{\mathbb{C}})^{\vee}) \xrightarrow{\simeq} \mathbb{Z}[\alpha_0,\beta_0]/\langle \alpha_0^n, \beta_0^n \rangle$$

$$K_0(\mathrm{Gr}(r,n)_{\mathbb{C}}) \to \mathbb{Z}[\alpha_0,\beta_0]/\langle \alpha_0^n, \beta_0^n \rangle, \quad c_M \cdot [\mathcal{O}(1)] \mapsto T_M(\alpha_0,\beta_0)$$

$$K_0^{\mathrm{mat}}(\mathbb{K}) \to \mathbb{Z}[\alpha_0,\beta_0]/\langle \alpha_0^n, \beta_0^n \rangle, \quad [M] \mapsto T_M(\alpha_0,\beta_0)$$

$$K_0^{\mathrm{mat}}(\mathbb{K}) \xrightarrow{\simeq} \mathbb{Z}[\alpha,\beta]$$

$$\mathbb{Z}[\alpha,\beta] \to \mathbb{Z}[\alpha_0,\beta_0]/\langle \alpha_0^n, \beta_0^n \rangle, \quad \alpha \mapsto \alpha_0, \ \beta \mapsto \beta_0$$

Can we recover Fink-Speyer's theorem?

$$\begin{array}{ccccc}
 & & K_0^{\mathrm{mat}}(\mathrm{Gr}(r,n)) & \xrightarrow{\det} & K_0(\mathrm{Gr}(r,n)_{\mathbb{C}}) \\
 & \overset{\pi^*}{\nearrow} \underset{\chi_M^*}{\swarrow} & \downarrow \pi_r^* & & \downarrow \pi_r^* \\
K_0^{\mathrm{mat}}(\mathbb{K}) & & K_0^{\mathrm{mat}}(\mathrm{Fl}(1,r,n-1;n)) & \xrightarrow{\det} & K_0(\mathrm{Fl}(1,r,n-1;n)_{\mathbb{C}}) \\
 & \overset{\pi^*}{\searrow} & \uparrow \pi_{1,n-1}^* & & \pi_{1,n-1}^* \uparrow \; \downarrow \pi_{1,n-1,*} \\
\simeq \downarrow & & K_0^{\mathrm{mat}}(\mathbb{P}^{n-1}\times(\mathbb{P}^{n-1})^\vee) & \xrightarrow{\det} & K_0(\mathbb{P}^{n-1}_{\mathbb{C}}\times(\mathbb{P}^{n-1}_{\mathbb{C}})^\vee) \\
 & & & & \downarrow \simeq \\
\mathbb{Z}[\alpha,\beta] & & \xrightarrow[\substack{\alpha\mapsto\alpha_0 \\ \beta\mapsto\beta_0}]{} & & \mathbb{Z}[\alpha_0,\beta_0]/\langle\alpha_0^n,\beta_0^n\rangle
\end{array}$$