The moduli space of matroids

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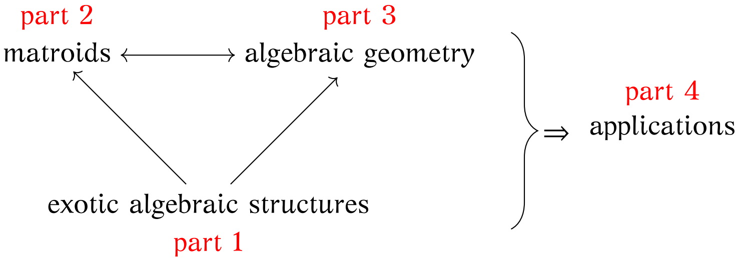
Based on joint work with Matthew Baker

Prologue

Content overview, and the thread



Theorem (Bland-Las Vergnas '78)

A matroid is regular if and only if it is binary and orientable.

Part 1: Bands

(joint with Baker and Jin)

Pointed monoids

A **pointed monoid** is a commutative semigroup A with 1 and 0, i.e. $1 \cdot a = a$ and $0 \cdot a = 0$ for all $a \in A$.

The **unit group** of A is

$$A^\times \;=\; \{a \in A \,|\, ab = 1 \text{ for some } b \in A\}.$$

The **ambient semiring** of A is

$$A^+ \;=\; \mathbb{N}[A]/\langle 0_A \sim 0_{\mathbb{N}[A]} \rangle \;=\; \{\textstyle\sum a_i \,|\, a_i \in A - \{0\}\}.$$

An **ideal** of A^+ is a subset I such that $0 \in I$, $\quad I + I = I$ and $A^+ \cdot I = I$.

Bands

A **band** is a pointed monoid B together with an ideal $N_B \subset B^+$ (the *nullset*) such that every $a \in B$ has a unique $-a \in B$ (its *additive inverse*) with $a + (-a) \in N_B$.

A **tract** is a band B with $B^\times = B - \{0\}$.

Example: A ring R defines the band $B = (R, \cdot)$ with nullset

$$N_B \;=\; \left\{ \sum a_i \,\middle|\, \sum a_i = 0 \text{ in } R \right\}.$$

If R is a field, then B is a tract. Further tracts:

$\mathbb{F}_2 = \{0, 1\}$	$N_{\mathbb{F}_2} = \{n.1 \mid n \text{ even}\}$	*field with 2 elements*
$\mathbb{K} = \{0, 1\}$	$N_{\mathbb{K}} = \{n.1 \mid n \neq 1\}$	*Krasner hyperfield*
$\mathbb{S} = \{0, \pm 1\}$	$N_{\mathbb{S}} = \{n.1 + m.(-1) \mid n = m = 0 \text{ or } n \neq 0 \neq m\}$	*sign hyperfield*
$\mathbb{F}_1^\pm = \{0, \pm 1\}$	$N_{\mathbb{F}_1^\pm} = \{n.1 + n.(-1) \mid n \geq 0\}$	*regular partial field*

Examples

A **band morphism** is a multiplicative map $f : B \to C$ with $f(0) = 0$ and $f(1) = 1$ such that $\sum f(a_i) \in N_C$ for all $\sum a_i \in N_B$.

Examples:

1. sign : $\mathbb{R} \to \mathbb{S}$ is a band morphism.
2. For every band B, there is a unique morphism $\mathbb{F}_1^{\pm} \to B$.
3. For every tract F, there is a unique morphism

$$t_F : F \longrightarrow \mathbb{K} \qquad \text{(the } \textit{terminal map}\text{)}$$

given by $t_F(a) = 1$ for all $a \neq 0$.

Part 2: Matroids

k-matroids

For the rest of the talk, let $0 \leq r \leq n$ be fixed integers and $E = \{1, \ldots, n\}$. Let k be a field.

Definition

A ***k*-matroid** (on E of rank r) is an r-dimensional subspace L of k^n.

Cryptomorphic description

A k-matroid is the same as a point of the Grassmannian $\mathrm{Gr}(r,n)(k)$, which is a subset of the projective space

$$\mathbb{P}^N(k) \;=\; \left\{ [\Delta_I] \,\middle|\, I \in \binom{E}{r} \right\}$$

where $N = \binom{n}{r} - 1$ and $\binom{E}{r}$ is the collection of all r-subsets of E.

k-matroids

In other words:

A k-matroid is a $k^\times$-class $[\Delta]$ of a *Grassmann-Plücker function*, which is a map

$$\Delta : \binom{E}{r} \longrightarrow k$$

with $\Delta_I \in k^\times$ for some I that satisfies the Plücker relations

$$\sum_{k=1}^{r+1} (-1)^k \, \Delta_{I \cup \{j_k\}} \, \Delta_{J - \{j_k\}} \;\; = \;\; 0$$

for all $I, J \subset E$ with $\#I = r - 1$, $J = \{j_1, \ldots, j_{r+1}\}$ where $j_1 < \ldots < j_{r+1}$ and $\Delta(I \cup \{j_k\}) = 0$ if $j_k \in I$.

Matroids

Definition

A **matroid** is a $\mathbb{K}$-matroid.

Example

Let k be a field and $[\Delta : \binom{E}{r} \to k] \in \mathrm{Gr}(r, n)(k)$ a k-matroid. Let $t_k : k \to \mathbb{K}$ be the terminal map. Then $M = [t_k \circ \Delta]$ is a matroid.

Definition

Let F be a tract with terminal map $t_F : F \to \mathbb{K}$. A matroid M is **representable over F** if there is a Grassmann-Plücker function $\Delta : \binom{E}{r} \to F$ such that $M = [t_F \circ \Delta]$.

Regular, binary and orientable matroids

A matroid M is called

- **regular** if M is representable over the regular partial field $\mathbb{F}_1^{\pm}$;
- **binary** if M is representable over the finite field $\mathbb{F}_2$;
- **orientable** if M is representable over the sign hyperfield $\mathbb{S}$.

Theorem (Bland-Las Vergnas '78)

A matroid is regular if and only if it is binary and orientable.

Part 3: Moduli spaces

Band schemes (joint with Baker and Jin)

An **affine band scheme** is a representable functor $\mathrm{Hom}(B, -) : \mathrm{Bands} \to \mathrm{Sets}$.

A **band scheme** is a functor $X : \mathrm{Bands} \to \mathrm{Sets}$ that has an *open cover* by affine band schemes.

A **morphism** of band schemes is a morphism of functors.

Example

1. The **projective n-space** $\mathbb{P}^n : \mathrm{Bands} \to \mathrm{Sets}$ is defined as

$$\mathbb{P}^n(B) \; = \; \{(a_0, \dots, a_n) \in B^{n+1} \, \big| \, a_i \in B^\times \text{ for some } i\} \, / \, B^\times.$$

2. The **Grassmannian** $\mathrm{Gr}(r, n) : \mathrm{Bands} \to \mathrm{Sets}$ is defined as

$$\mathrm{Gr}(r, n)(B) \; = \; \{[\Delta : \tbinom{E}{r} \to B] \, \big| \, \text{Plücker relations}\} \; \subset \; \mathbb{P}^{\binom{n}{r}-1}(B).$$

The moduli space of matroids

Let X be a band scheme. A **matroid bundle** on X is an isomorphism class of a *Grassmann-Plücker function* $\Delta : \binom{E}{r} \to \Gamma(X, \mathcal{L})$ where $\mathcal{L}$ is a *line bundle* on X.

Theorem (Baker-L '21)

1. *For every band B and $X = \mathrm{Hom}(B, -)$, there is a canonical bijection*

$$\Phi_B : \{B\text{-}matroids\} \longrightarrow \{matroid\ bundles\ on\ X\}.$$

2. *For every band scheme X, there is a canonical bijection*

$$\Psi_X : \mathrm{Hom}\,(X, \mathrm{Gr}(r,n)) \longrightarrow \left\{ \begin{array}{c} matroid\ bundles\ on\ X \\ of\ rank\ r\ on\ \{1, \dots, n\} \end{array} \right\}.$$

The **universal family** on $\mathrm{Gr}(r, n)$ is $\mathcal{M}^{\mathrm{univ}} = \Psi_{\mathrm{Gr}(r,n)}(\mathrm{id}_{\mathrm{Gr}(r,n)})$.
The matroid bundle $\Psi_X(\varphi : X \to \mathrm{Gr}(r, n))$ is the *pullback* $\varphi^* \mathcal{M}^{\mathrm{univ}}$.

Part 4: Foundations of matroids

The universal tract

In the case $X = \mathrm{Hom}(\mathbb{K}, -)$, we obtain a bijection

$$\Psi_X : \quad \mathrm{Gr}(r,n)(\mathbb{K}) \quad \longleftrightarrow \quad \{\text{matroids of rank } r \text{ on } E\}.$$

Given a matroid M, we denote $\Psi_X^{-1}(M)$ by $\chi_M : \operatorname{Spec}\mathbb{K} \to \mathrm{Gr}(r,n)$. Let $x_M \in \mathrm{Gr}(r,n)$ be the image point of χ_M.

The **universal tract** of M is the *residue field* T_M of $\mathrm{Gr}(r,n)$ at x_M. It is indeed a tract.

First application: thin Schubert cells

Let M be a matroid and k a field with terminal map $t_k : k \to \mathbb{K}$. The **thin Schubert cell** of M over k is

$$\mathcal{X}_M(k) \;=\; \left\{ [\Delta] \in \mathrm{Gr}(r,n)(k) \,\middle|\, M = [t_k \circ \Delta] \right\}.$$

Universality theorem (Mnëv / Sturmfels / Lafforgue / Vakil)

$\mathcal{X}_M(k)$ can be arbitrarily complicated (for fixed k and varying M).

Theorem (Baker-L '18)

Let T_M be the universal tract of M. Then there exists a canonical bijection $\mathcal{X}_M(k) \to \mathrm{Hom}(T_M, k)$.

Corollary

M is representable over k $\Leftrightarrow$ there is a morphism $T_M \to k$.

An important observation

Many tracts have the following property:

A map $\Delta : \binom{E}{r} \to F$, for which $M = [t_F \circ \Delta]$ is a matroid, is a Grassmann-Plücker function if it satisfies the *3-term Plücker relations*

$$\Delta_{Iab} \cdot \Delta_{Icd} \; - \; \Delta_{Iac} \cdot \Delta_{Ibd} \; + \; \Delta_{Iad} \cdot \Delta_{Ibc} \; \in \; N_F$$

for all $I \in \binom{E}{r-2}$ and $a < b < c < d$ in E.

For the purpose of this talk, we call a tract with this property a **perfect tract**.

Examples of perfect tracts are fields, $\mathbb{K}$, $\mathbb{S}$ and $\mathbb{F}_1^{\pm}$.

The foundation

A **pasture** is a tract T whose null set is generated by 3-term relations: $N_T = \langle a+b+c \mid a+b+c \in N_T \rangle$.

The **universal pasture** of M is the 3-term truncation

$$P_M \;=\; \left\langle a+b+c \,\middle|\, a+b+c \in N_{T_M} \right\rangle$$

of the universal tract T_M.

The **foundation** of M is the subpasture F_M of P_M generated by its *cross ratios*, which are terms of the form

$$\frac{\Delta_{Iab} \cdot \Delta_{Icd}}{\Delta_{Iac} \cdot \Delta_{Ibd}}.$$

Application: Bland-Las Vergnas's theorem

Theorem (Baker-L '18)

Let M be a matroid with foundation F_M and F a perfect tract. Then M is representable over F if and only if there is a morphism $F_M \to F$.

Theorem (Baker-L '18)

A matroid M is

- *regular if and only if $F_M = \mathbb{F}_1^{\pm}$;*
- *binary if and only if $F_M = \mathbb{F}_1^{\pm}$ or $\mathbb{F}_2$.*

Theorem (Bland-Las Vergnas '78)

A matroid M is regular if and only if it is binary and orientable.

Proof.

M is binary and orientable $\Leftrightarrow$ $F_M = \mathbb{F}_1^{\pm}$ or $\mathbb{F}_2$, and M is orientable $\Leftrightarrow$ $F_M = \mathbb{F}_1^{\pm}$ (there is no morphism $\mathbb{F}_2 \to \mathbb{S}$) $\Leftrightarrow$ M is regular $\square$