Moduli spaces in matroid theory

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Based on joint work
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Prologue

Spaces of matroids

Let $E = \{1, \ldots, n\}$ and $0 \leq r \leq n$.

Combinatorial flag varieties ([Borovic-Gelfand-White '00])

$$\Omega_{S_n} \;=\; \{\text{flag matroids on } E\},$$

where a flag matroid $\mathbf{M} = (M_1, \ldots, M_{k+1})$ defines a k-simplex.

MacPhersonians (after [Mnev-Ziegler '93])

$$\mathrm{MacPh}(r, E) \;=\; \{\text{oriented matroids of rank } r \text{ on } E\},$$

which gain a topology / simplicial structure from weak maps.

Dressians (after [Speyer '08])

$$\mathrm{Dr}(r, E) \;=\; \{\text{valuated matroids of rank } r \text{ on } E\},$$

considered as tropical varieties.

Some exotic algebraic objects

The **Krasner hyperfield** is $\mathbb{K} = \{0, 1\}$ together with the obvious multiplication and the addition table

+	0	1
0	0	1
1	1	0, 1

The **sign hyperfield** is $\mathbb{S} = \{0, 1, -1\}$ together with the obvious multiplication and the addition table

+	0	1	−1
0	0	1	−1
1	1	1	0, ±1
−1	−1	0, ±1	−1

The **tropical hyperfield** is $\mathbb{T} = \mathbb{R}_{\geq 0}$ together with the obvious multiplication and the addition

$$a \boxplus b = \begin{cases} \{\max\{a, b\}\} & \text{if } a \neq b; \\ [0, a] & \text{if } a = b. \end{cases}$$

The **regular partial field** is $\mathbb{F}_1^{\pm} = \{0, 1, -1\}$ together with the obvious multiplication and the addition table

+	0	1	−1
0	0	1	−1
1	1	−	0
−1	−1	0	−

Part 1. Bands (joint with Baker)

Pointed monoids

A **pointed monoid** is a commutative semigroup A with 1 and 0, i.e. $1 \cdot a = 1$ and $0 \cdot a = 0$ for all $a \in A$.

The **unit group** of A is

$$A^{\times} \;=\; \{a \in A \,|\, ab = 1 \text{ for some } b \in A\},$$

and the **ambient semiring** of A is

$$A^{+} \;=\; \mathbb{N}[A]/\langle 0_A \sim 0_{\mathbb{N}[A]} \rangle \;=\; \{\textstyle\sum a_i \,|\, a_i \in A - \{0\}\}$$

An **ideal** of A^+ is a subset I such that $0 \in I$, $\quad A^+ \cdot I = I$ and $I + I = I$.

Bands

A **band** is a pointed monoid B together with an ideal $N_B \subset B^+$ (the *nullset*) and together with an element $-1 \in B$ (the *negative one*) such that $(-1)^2 = 1$ and $1 - 1 := 1 + (-1)$ is in N_B.

A **band morphism** is a map $f : B \to C$ such that

1. $f(0) = 0,\ f(1) = 1$ and $f(-1) = -1$;
2. $f(a \cdot b) = f(a) \cdot f(b)$;
3. $\sum a_i \in N_B$ implies $\sum f(a_i) \in N_C$

for all $a, b, a_i \in B$.

Remark: The category of bands has all limits and colimits, free algebras and “quotients”.

An **idyll** is a band B such that $B^\times = B - \{0\}$ and such that $a + b \in N_B$ if and only if a equals $-b := (-1) \cdot b$.

Examples

- A ring R defines the band $B = (R, \cdot)$ with nullset

$$N_B = \left\{ \sum a_i \,\middle|\, \sum a_i = 0 \text{ in } R \right\}$$

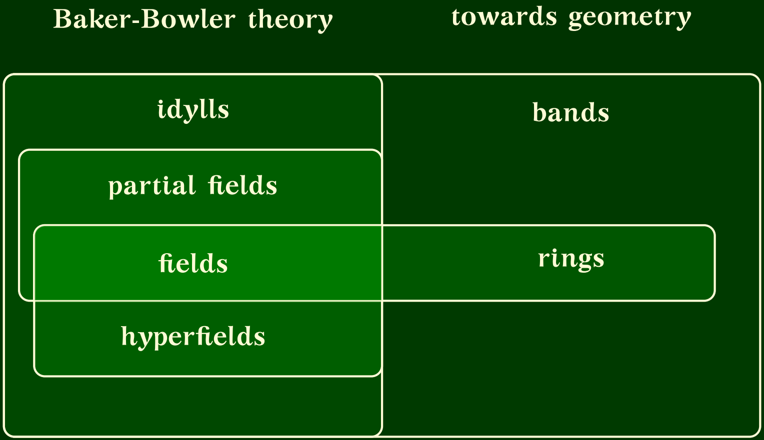
  and negative one -1. If R is a field, then B is an idyll.
- A hyperfield F defines the idyll $B = (F, \cdot)$ with nullset

$$N_B = \left\{ \sum a_i \,\middle|\, 0 \in \boxplus\, a_i \text{ in } F \right\}$$

  and negative one -1. For instance,

$\mathbb{K} = \{0, 1\}$	$N_{\mathbb{K}} = \{n.1 \mid n \neq 1\}$ *(terminal as idyll)*
$\mathbb{S} = \{0, \pm 1\}$	$N_{\mathbb{S}} = \{n.1 + m.(-1) \mid n = m = 0 \text{ or } n \neq 0 \neq m\}$
$\mathbb{T} = \mathbb{R}_{\geq 0}$	$N_{\mathbb{T}} = \{\sum a_i \mid \text{maximum occurs twice}\}$
$\mathbb{F}_1^{\pm} = \{0, \pm 1\}$	$N_{\mathbb{F}_1^{\pm}} = \{n.1 + n.(-1) \mid n \geq 0\}$ *(initial as both band and idyll)*

Categorical landscape



Part 2. Baker-Bowler theory

B-matroids

Let $E = \{1, \dots, n\}$ and $0 \le r \le n$. Let B be an idyll.

A **Grassmann-Plücker function** (of rank r on E) in B is a non-trivial and alternating map $\Delta : E^r \to B$ that satisfies the *Plücker relations*

$$\sum_{k=1}^{r+1} (-1)^k \, \Delta(e_1, \dots, e_{r-1}, f_k) \, \Delta(f_1, \dots, \widehat{f_k}, \dots, f_{r+1}) \ \in \ N_B$$

for all $e_1, \dots, e_{r-1}, f_1, \dots, f_{r+1} \in E$.

A ***B*-matroid** (of rank r on E) is a $B^\times$-class $M = [\Delta]$ of a Grassmann-Plücker function $\Delta : E^r \to B$.

Baker-Bowler theory

Example

1. A matroid M of rank r on E corresponds to the $\mathbb{K}$-matroid $[\Delta : E^r \to \mathbb{K}]$ with
$$\Delta(e_1, \ldots, e_r) \;=\; \begin{cases} 1 & \text{if } \{e_1, \ldots, e_r\} \text{ is a basis of } M; \\ 0 & \text{if not.} \end{cases}$$
2. Oriented matroids correspond to $\mathbb{S}$-matroids.
3. Valuated matroids correspond to $\mathbb{T}$-matroids.

Theorem (Baker-Bowler '19)

- *Cycles:* $\mathcal{C}(M) \subset B^n$*, with cryptomorphic axioms*
- *Duality:* M^*
- *Orthogonality:* $\mathcal{C}(M) \perp \mathcal{C}(M^*)$
- *Vectors:* $\mathcal{V}(M) = \mathcal{C}(M^*)^\perp$*, axioms by* [Anderson '19]
- *Minors:* $M \backslash I / J$

Part 3. Band schemes (joint with Baker)

Localizations

Let B be a band. A **multiplicative set** in B is a subset S of B such that $1 \in S$ and $S \cdot S = S$.

The **localization** of B at S is the monoid

$$S^{-1}B \;=\; \left\{ \tfrac{a}{s} \;\middle|\; a \in B, s \in S \right\},$$

together with the nullset

$$S^{-1}N_B \;=\; \left\langle \textstyle\sum \tfrac{a_i}{1} \;\middle|\; \textstyle\sum a_i \in N_B \right\rangle$$

and the negative one $\frac{-1}{1}$.

A **prime ideal** of B is a subset $\mathfrak{p}$ of B such that $0 \in \mathfrak{p}$, $B \cdot \mathfrak{p} = \mathfrak{p}$ and $S = B - \mathfrak{p}$ is a multiplicative set in B.

In particular:

$$B[h^{-1}] = S^{-1}B \qquad \text{for } h \in B \text{ and } S = \{h^i\}_{i \geq 0};$$

$$B_{\mathfrak{p}} = S^{-1}B \qquad \text{for } \mathfrak{p} \subset B \text{ prime and } S = B - \mathfrak{p}.$$

The spectrum

Let B be a band. The **spectrum of B** is the set

$$\operatorname{Spec} B \;=\; \{\text{prime ideals } \mathfrak{p} \subset B\},$$

together with the topology generated by the *principal opens*

$$U_h \;=\; \{\mathfrak{p} \subset B \mid h \notin \mathfrak{p}\} \qquad\qquad (\text{for } h \in B).$$

Theorem

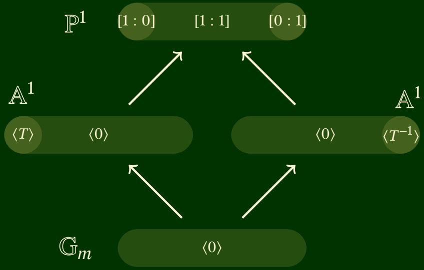
There exists a uniquely determined sheaf $\mathcal{O}_X$ in bands on $X = \operatorname{Spec} B$ (the structure sheaf) such that

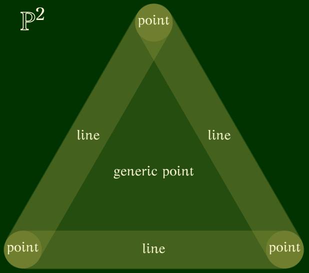
- $\mathcal{O}_X(U_h) = B[h^{-1}]$ *for all* $h \in B$*;*
- $\mathcal{O}_{X,\mathfrak{p}} = B_\mathfrak{p}$ *for all* $\mathfrak{p} \in X$*.*

A **band scheme** is a topological space X together with a sheaf $\mathcal{O}_X$ in bands that is locally isomorphic to the spectra of bands.

There is a natural notion of morphisms of band schemes.

Examples of band schemes (over $\mathbb{F}_1^{\pm}$)





Similarly, there are projective spaces $\mathbb{P}^n$ for all $n > 0$ over $\mathbb{F}_1^{\pm}$.

More generally, for every fan Δ of cones in $\mathbb{R}^n$, there is a *toric band scheme* $\Sigma(\Delta)$ over $\mathbb{F}_1^{\pm}$.

Base extension to $\mathbb{Z}$ and descent to $\mathbb{F}_1^{\pm}$

Let B be a band. The **base extension** of B to $\mathbb{Z}$ is the ring

$$B_{\mathbb{Z}}^{+} \;=\; \mathbb{Z}[B]/\langle N_B\rangle.$$

This extends to the base extension for band schemes:

$$X \;=\; \bigcup \operatorname{Spec} B_i \;\longmapsto\; X_{\mathbb{Z}}^{+} \;=\; \bigcup \operatorname{Spec} B_{i,\mathbb{Z}}^{+}.$$

Conversely, the closed immersion $X \to Y_{\mathbb{Z}}^{+}$ of a scheme X into the base extension of a band scheme Y defines a **band model** $\mathcal{X}$ of X together with a closed immersion $\mathcal{X} \to Y$.

Part 4. Moduli spaces of matroids (joint with Baker)

Matroid bundles and their moduli space

The *Plücker embedding* $\mathrm{Gr}(r,n)_{\mathbb{Z}} \hookrightarrow \mathbb{P}^N_{\mathbb{Z}}$ (for $N = \binom{n}{r} - 1$) defines the band model $\mathrm{Gr}(r,n) \hookrightarrow \mathbb{P}^N$.

There is a natural extension of B-matroids to *matroid bundles* over a band scheme X (in terms of Grassmann-Plücker functions), which poses the *moduli functor*

$$X \longmapsto \{\text{matroid bundles on } X\}.$$

Theorem (Baker-L '21)

$\mathrm{Gr}(r,n)$ *is the fine moduli space of matroid bundles.*
As a consequence, there is a canonical bijection

$$\mathrm{Gr}(r,n)(B) \longrightarrow \{B\text{-}matroids\}$$

for every idyll B.

Part 5. Flag matroids with coefficients (joint with Jarra)

Flag B-matroids

Let $0 \le r_1 \le \cdots \le r_s \le n$ and $\mathbf{r} = (r_1, \ldots, r_s)$. Let $\mathrm{Fl}(\mathbf{r}, n)_{\mathbb{Z}}$ be the flag variety of type $\mathbf{r}$ flags in n-space. The closed immersion

$$\eta_{\mathbb{Z}} : \ \mathrm{Fl}(\mathbf{r}, n)_{\mathbb{Z}} \ \longrightarrow \ \prod_{i=1}^{s} \ \mathrm{Gr}(r_i, n)_{\mathbb{Z}}$$

defines the band model

$$\eta : \ \mathrm{Fl}(\mathbf{r}, n) \ \longrightarrow \ \prod_{i=1}^{s} \ \mathrm{Gr}(r_i, n).$$

Definition

Let B be an idyll. A **flag B-matroid** is a sequence $(M_1, \ldots, M_s)$ of B-matroids that corresponds to a point in the image of

$$\eta_B : \mathrm{Fl}(\mathbf{r}, n)(B) \ \longrightarrow \ \prod_{i=1}^{s} \ \mathrm{Gr}(r_i, n)(B).$$

Example

Flag matroids correspond to flag $\mathbb{K}$-matroids.
Valuated flag matroids (after [Brandt-Eur-Zhang '21]) correspond to flag $\mathbb{T}$-matroids.

Baker-Bowler theory for flag matroids

Theorem (Jarra-L '22)

Let B be an idyll and $\mathbf{M} = (M_1, \dots, M_s)$ a sequence of B-matroids of ranks $\mathbf{r} = (r_1, \dots, r_s)$ on E. Then we have:

- *Cryptomorphic descriptions:*
  1. $\mathbf{M} \in \mathrm{Fl}(\mathbf{r}, E)(B)$
  2. $\mathcal{C}(M_i^*) \subset \mathcal{V}(M_j^*)$ *for all* $1 \le i \le j \le s$
  3. *given* $M_i = [\Delta_i]$ *for* $i = 1, \dots, s$, *we have*

$$\sum_{k=1}^{r_j+1} (-1)^k \, \Delta_i(e_1, \dots, e_{r_i-1}, f_k) \, \Delta_j(f_1, \dots, \widehat{f_k}, \dots, f_{r_j+1})$$

     for all $i \le j$ *and* $e_1, \dots, e_{r_i-1}, f_1, \dots, f_{r_j+1} \in E$
- *Duality:* $\mathbf{M}^* = (M_s^*, \dots, M_1^*)$
- *Orthonality:* $\mathcal{C}(M_i^*) \perp \mathcal{C}(M_j)$ *for* $i \le j$
- *Minors:* $\mathbf{M} \backslash I / J = (M_1 \backslash I/J, \dots, M_s \backslash I/J)$

Part 6. Applications

Application 1 (with Baker)

A (usual) matroid M corresponds to a $\mathbb{K}$-rational point of $\mathrm{Gr}(r,n)$, which comes with a "residue field" k_M.
The **foundation** of M is an idyll F_M derived from k_M.

Theorem (Baker-L '21)

M is representable over a (partial) field K (resp. is orientable) if and only if there exists a morphism $F_M \to K$ (resp. $F_M \to \mathbb{S}$).

Further results:

1. Structure theorem for F_M of M without minors of types $U(2,5)$ and $U(3,5)$, with manifold consequences.
   Sample application: let $\mathcal{X}_M(K)$ be the variety of rescaling classes of M over a field K. Then for all ternary M,

$$\#\,\mathcal{X}_M(\mathbb{F}_4)\cdot\#\,\mathcal{X}_M(\mathbb{F}_5) \;=\; \#\,\mathcal{X}_M(\mathbb{F}_8).$$

2. Computational methods for F_M, Macaulay2 package by Chen and Zhang.

Application 2 (with Jarra)

We gain the notion of a strong map $f : M \to N$ of B-matroids in terms of the cryptomorphic descriptions:

1. $f(C(M)) \subset \mathcal{V}(N)$;
2. for all $e_1, \ldots, e_{\mathrm{rk}(M)-1} \in E_M$ and $f_1, \ldots, f_{\mathrm{rk}(N)+1} \in E_N$, we have

$$\sum_{k=1}^{\mathrm{rk}(N)+1} (-1)^k \, \Delta_M(e_1, \ldots, e_{\mathrm{rk}(M)-1}, f_k) \cdot \Delta_N(f_1, \ldots, \widehat{f_k}, \ldots, f_{\mathrm{rk}(N)+1}) \ \in \ N_B.$$

Problem: Strong maps are in general not composable.
Thus we might need to impose additional requirements.

Application 3 (with Baker / Jarra)

Let B be a band with topology, e.g.

- $\mathbb{R}$ with the usual real topology;
- $\mathbb{T} = \mathbb{R}_{\geq 0}$ with the real topology;
- $\mathbb{S}$ with the topology generated by $\{1\}$ and $\{-1\}$.

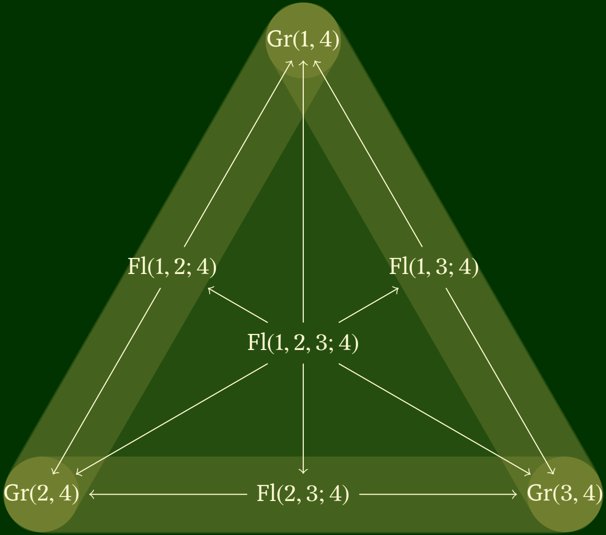
Then $\mathrm{Fl}(\mathbf{r}, E)(B)$ inherits the *fine topology* (after [L-Salgado '16]).

We get:

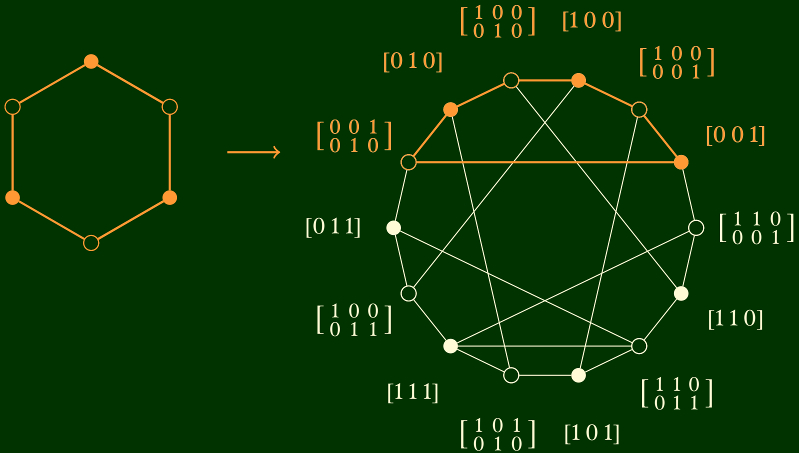
- $\mathrm{Fl}(\mathbf{r}, n)(\mathbb{R})$ is the real flag manifold (of type $\mathbf{r}$ flags in $\mathbb{R}^n$).
- $\mathrm{Gr}(r, n)(\mathbb{T})$ is the Dressian (of tropical linear spaces in $\mathbb{T}^n$).
- $\mathrm{Gr}(r, n)(\mathbb{S})$ is the MacPhersonian (of oriented matroids).
- $\Omega_{S_n} = \coprod_{\mathbf{r}} \mathrm{Fl}(\mathbf{r}, n)(\mathbb{K})$, with face relation by omitting entries:

$$\mathrm{Gr}(1,3) \longleftarrow \mathrm{Fl}(1,2;3) \longrightarrow \mathrm{Gr}(2,3)$$

Face relations for flags on $E = \{1, \ldots, 4\}$



Revisited: the combinatorial flag variety Ω_{S_3}



Coxeter complex of S_3
$\widehat{=}$ closed points of $\coprod_{\mathbf{r}} \mathrm{Fl}(\mathbf{r}, E)$

$\Omega_{S_3} \widehat{=} \mathbb{K}$-points of $\coprod_{\mathbf{r}} \mathrm{Fl}(\mathbf{r}, E)$

Excerpt from “Coxeter matroids” by Borovik, Gelfand and White

Many geometries over fields have formal analogues which can be thought of as geometries over the field of 1 *element.*

[...]

In general, the Coxeter complex $\mathcal{W}$ *of a Coxeter group* W *is a thin building of type* W *and behaves like the building of type* W *over the field of* 1 *element.*

However, the Coxeter complex has a relatively poor structure. In many aspects, Ω_W *and* Ω^*_W *are more suitable candidates for the role of a “universal”* combinatorial *geometry of type* W *over the field of* 1 *element.*