What is a tropical scheme?

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Part 1: What is a tropical variety?

Tropical curves

A **tropical curve (in $\mathbb{R}^n$)** is an embedded graph Γ in $\mathbb{R}^n$ with possibly unbounded edges together with a weight function

$$m : \text{ Edge}\,\Gamma \longrightarrow \mathbb{Z}_{>0}$$

such that all edges are lines with rational slopes and such that the following condition is satisfied for every vertex p of Γ:

for every edge e containing p, let $v_e \in \mathbb{Z}^n$ be the smallest nonzero vector pointing from p in the direction of e (*primitive vector*); then

$$\sum_{p \in e} m(e) \cdot v_e = 0 \qquad\qquad (\textit{balancing condition}).$$

Polyhedral complexes

A **polyhedron** P **(in $\mathbb{R}^n$)** is an intersection of finitely many halfspaces of $\mathbb{R}^n$, which are subsets of the form

$$H = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, a_1 x_1 + \cdots + a_n x_n \geq b \right\}$$

for some $a_1, \ldots, a_n, b \in \mathbb{R}$.

A **face** of a polyhedron P is a nonempty intersection of P with a halfspace H such that the boundary of H does not contain interior points of P. Note that each face is itself a polyhedron.

A **polyhedral complex (in $\mathbb{R}^n$)** is a finite collection Δ of polyehdra in $\mathbb{R}^n$ such that

1. each face of a polyhedron in Δ is in Δ;
2. the intersection of two polyhedra in Δ is a face of both polyhedra or empty.

Polyhedral complexes

Let Δ be a polyhedral complex. The **support** of Δ is

$$|\Delta| \;=\; \bigcup_{P \in \Delta} P.$$

The **dimension** of Δ is

$$\dim \Delta \;=\; \max\{\,\dim P \mid P \in \Delta\,\}.$$

Δ is **equidimensional** if

$$|\Delta| \;=\; \bigcup_{\dim P = \dim \Delta} P.$$

Δ is **rational** if every polyhedron P in Δ is the intersection of halfspaces

$$H \;=\; \{\,(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1 x_1 + \cdots + a_n x_n \geq b\,\}$$

with $a_1, \ldots, a_n \in \mathbb{Q}$.

Tropical varieties

A **tropical variety (in $\mathbb{R}^n$)** is an equidimensional and rational polyhedral complex Δ together with a weight function

$$m : \big\{ P \in \Delta \,\big|\, \dim P = \dim \Delta \big\} \longrightarrow \mathbb{Z}_{>0}$$

such that ... for every polyhedron $P \in \Delta$ with $\dim P = \dim \Delta - 1$, the top dimensional polyhedra in Δ containing P satisfy the balancing modulo the affine linear space spanned by P.

Part 2: What is the tropicalization of a classical variety?

Nonarchimedean fields

A **nonarchimedean field** is an algebraically closed field k together with a nontrivial *nonarchimedean absolute value*, which is a function

$$v : k \longrightarrow \mathbb{R}_{\geq 0}$$

with dense image such that for all $a, b \in k$,

- $v(0) = 0$ and $v(1) = 1$;
- $v(ab) = v(a)v(b)$;
- $v(a + b) \leq \max\{v(a), v(b)\}$.

Tropicalizations

Let $X \subset (k^\times)^n$ be an algebraic variety, i.e. the zero set of Laurent polynomials

$$f_1, \ldots, f_r \in k[T_1^{\pm 1}, \ldots, T_n^{\pm 1}].$$

Consider the map

$$\mathrm{trop} : (k^\times)^n \xrightarrow{(v,\ldots,v)} \mathbb{R}_{>0}^n \xrightarrow{(\log,\ldots,\log)} \mathbb{R}^n.$$

The **tropicalization of X** is defined as the topological closure

$$X^{\mathrm{trop}} = \overline{\mathrm{trop}(X)}$$

of the image of X in $\mathbb{R}^n$.

Fundamental theorems of tropical geometry

Theorem

Let (k, v) be a nonarchimedean field and $X \subset (k^\times)^n$ an equidimensional algebraic variety. Then ...

1. $X^{\mathrm{trop}} = |\Delta|$ *for a rational and equidimensional polyhedral complex Δ;*

2. $X \subset (k^\times)^n$ *determines a weight function*

$$m : \left\{ P \in \Delta \,\middle|\, \dim P = \dim \Delta \right\} \longrightarrow \mathbb{Z}_{>0}$$

such that (Δ, m) is a tropical variety.

Part 3: Two problems with the concept of a tropical variety

The first problem

The polyhedral complex Δ with $|\Delta| = X^{\text{trop}}$ is not determined by the classical variety $X \subset (k^\times)^n$. In other words:

**the tropicalization of a classical variety
is not a tropical variety.**

The tropical numbers

The **tropical semifield** is the set $\mathbb{T} = \mathbb{R}_{\geq 0}$ together with the addition

$$a \oplus b = \max\{a, b\}$$

and the usual multiplication

$$a \odot b = ab$$

of nonnegative real numbers a, b.

$\left[\begin{array}{c} \text{because addition} \\ \text{is not invertible} \end{array}\right]$

Remark

Together with these operations $\mathbb{T}$ is indeed a semisemifield.

A nonarchimedean valuation is the same as a multiplicative map $v : k \to \mathbb{T}$ that is *subadditive*, i.e. $v(a + b) \leq v(a) \oplus v(b)$.

The map $\log : \mathbb{T} \to \mathbb{R} \cup \{-\infty\}$ defines an isomorphism between the tropical semifield $\mathbb{T}$ and the *max-plus-algebra* $(\mathbb{R} \cup \{-\infty\}, \max, +)$.

Tropical polynomials as tropical functions

The **tropical polynomial algebra in** $T_1, \ldots, T_n$ is the set

$$\mathbb{T}[T_1, \ldots, T_n] = \left\{ \sum_{J=(e_1, \ldots, e_n)} a_J T_1^{e_1} \cdots T_n^{e_n} \ \middle| \ \begin{array}{c} a_J \in \mathbb{T}, \\ \text{almost all} = 0 \end{array} \right\},$$

which is a semiring with respect to the usual addition and multiplication rules for polynomials.

A tropical polynomial $f = \sum a_J T_1^{e_1} \cdots T_n^{e_n}$ defines the function

$$f(-): \qquad \mathbb{T}^n \qquad \longrightarrow \qquad \mathbb{T}.$$

$$x = (x_1, \ldots, x_n) \ \longmapsto \ f(x) = \max\left\{ a_J x_1^{e_1} \cdots x_n^{e_n} \right\}$$

The second problem

Different polynomials can define the same function.

In other words:

tropical functions are not tropical polynomials.

Hilbert's Nullstellensatz guarantees that for varieties over an algebraically closed field, functions are the same as polynomials.

However, if one tries to generalize the concept of a variety to arbitrary field, or even rings, one faces the same problem:

functions are *not* polynomials.

Grothendieck invention of a *scheme* surpasses this problem.

The proposed solution to these problems

Use *semiring schemes*!

Such objects have been defined as *by-products* of $\mathbb{F}_1$-geometry (cf. the works of Durov, Toën-Vaquié and the speaker).

Part 4: What is a semiring scheme?

Grothendieck's schemes

Let k be an algebraically closed field and $X \subset k^n$ a variety, i.e. the zero set of polynomials $f_1, \ldots, f_r \in k[T_1, \ldots, T_n]$.

The **ideal of definition** of X is

$$I = \left\{ f \in k[T_1, \ldots, T_n] \;\middle|\; \begin{array}{c} f(x_1, \ldots, x_n) = 0 \\ \text{for all } (x_1, \ldots, x_n) \in X \end{array} \right\}.$$

The **ring of regular functions** of X is

$$A = k[T_1, \ldots, T_n]/I.$$

The associated scheme or **spectrum** of A is the set

$$\mathrm{Spec}\, A = \{\, \text{prime ideals } \mathfrak{p} \text{ of } A \,\}$$

together with the topology generated by the *principal open subsets*

$$U_h = \{\, \mathfrak{p} \subset A \mid h \notin \mathfrak{p} \,\}$$

for $h \in A$ and with the *structure sheaf*

$$\mathcal{O} : \; \{\text{open subsets of } \mathrm{Spec}\, A\} \; \longrightarrow \; \text{Rings.}$$
$$U_h \qquad\qquad \longmapsto \quad A[h^{-1}]$$

Recovering the variety

We can recover the ring of regular functions $A = k[T_1, \ldots, T_n]/I$ as

$$\mathcal{O}(\operatorname{Spec} A) \;=\; A[1^{-1}] \;=\; A.$$

We can recover the original variety X via the canonical bijection

$$X \longrightarrow \operatorname{Hom}_k(A, k) \;=\; \text{``}\operatorname{Hom}_k(\operatorname{Spec} k, \operatorname{Spec} A)\text{''}$$

that sends a point $x = (x_1, \ldots, x_n)$ of X to the evaluation map

$$\operatorname{ev}_x : \; h \; \mapsto \; h(x).$$

Its inverse sends a homomorphism $f : A \to k$ to the point $\big(f(T_1), \ldots, f(T_n)\big)$ of X.

Semiring schemes

The definition of $\text{Spec}\,A$ extends to any ring A.
In fact, it extends to any semiring A.

The spectrum of A is the set

$$\text{Spec}\,A \ = \ \{\,\text{prime ideals }\mathfrak{p}\text{ of }A\,\}$$

together with the topology generated by the principal open subsets

$$U_h \ = \ \{\,\mathfrak{p} \subset A \,|\, h \notin \mathfrak{p}\,\}$$

for $h \in A$ and with the structure sheaf

$$\mathcal{O}: \quad \{\text{open subsets of } \text{Spec}\,A\} \quad \longrightarrow \quad \text{Rings} \ \ \text{Semirings}$$
$$U_h \qquad\qquad \longmapsto \qquad A[h^{-1}]$$

Part 5: Scheme theoretic tropicalization

Some notation

For a multi-index $J = (e_1, \ldots, e_n)$, we write $T^J = T_1^{e_1} \cdots T_n^{e_n}$ and $x^J = x_1^{e_1} \cdots x_n^{e_n}$.

Let (k, v) be a nonarchimedean field and $x \in k^n$. Let $f = \sum a_J T^J \in k[T_1, \ldots, T_n]$. We define

$$f^{\text{trop}} = \sum v(a_J) T^J \qquad \in \mathbb{T}[T_1, \ldots, T_n].$$

The Giansiracusa tropicalization

Let (k, v) be a nonarchimedean field and $X \subset k^n$ a variety with ideal of definition I.

The **Giansiracusa tropicalization** of X is the semiring scheme

$$\mathsf{Trop}_v(X) \ = \ \mathsf{Spec}\left(\mathbb{T}[T_1, \ldots, T_n] \,/\, \mathsf{bend}_v(I)\right)$$

where the *bend relations* are defined as

$$\mathsf{bend}_v(I) \ = \ \left(f^{\mathsf{trop}} \sim f^{\mathsf{trop}} + v(b_J)\, T^J \,\Big|\, f + b_J T^J \in I \right).$$

Recovering the tropical variety

Theorem (Jeffrey and Noah Giansiracusa '13)

We can recover the tropical variety X^{trop} as a set via a natural bijection

$$X^{\text{trop}} \xrightarrow{\sim} \text{Hom}_{\mathbb{T}}(\text{Spec}\,\mathbb{T}, \text{Trop}_v(X)).$$

Theorem (Maclagan-Rincón '14)

Assume that $X \subset (k^\times)^n$ is equidimensional. Then the weight function m of any realization of X^{trop} as a tropical variety (Δ, m) is determined by the embedding of $\text{Trop}_v(X)$ into the n-dimensional tropical torus "$\mathbb{R}^n$".

Theorem (L. '15)

Refinement and generalization of the Giansiracusa tropicalization in terms of "blue schemes", which applies to (some) skeletons of Berkovich spaces.

New problems

There are too many $\mathbb{T}$-schemes!

Not for all $\mathbb{T}$-schemes X, the set $X(\mathbb{T}) = \mathrm{Hom}_{\mathbb{T}}(\mathrm{Spec}\,\mathbb{T}, X)$ is the support of a polyhedral complex Δ.

And even if so, Δ might not be balanced (w.r.t. any weight function) in general.

It is not even clear yet how to express the balancing condition.

A new hope

Idea (Maclagan-Rincón '16)
Use matroid conditions to define "tropical schemes".

Drawback
Taking Maclagan-Rincón's definition of a tropical scheme literal leads to difficulties with primary decompositions (important for weights) and intersection theory.

What is a tropical scheme?